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IDENTIFICATION OF DEGENERATE TIME- AND SPACE-DEPENDENT KERNELS IN HEAT FLOW

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Abstract -An inverse problem to determine degenerate time- and space-dependent relaxation kernels of internal energy and heat flux by means of temperature measurements is considered. Existence and uniqueness of a solution to the inverse problem are proved.

1. INTRODUCTION

Constitutive relations in the theory of heat conduction in materials with thermal memory contain time-dependent (and in the case of non-homogeneity also space-dependent) memory (or relaxation) kernels [1-4]. These kernels are often unknown in the practice. To determine them, inverse problems are used.

Various problems to identify time-dependent memory kernels in heat conduction have been studied in a number of papers (see [5-8] and references therein). When the kernels are both time- and space-dependent, inverse problems based on restricted Dirichlet-to-Neumann map in general case [9] and single trace measurements in stratified cases [10] are in the use.

In some context the kernels can degenerate, i.e. be represented as finite sums of products of known space-dependent functions times unknown time-dependent coefficients. This is so when either the medium is piecewise continuous or a problem for a general kernel is replaced by a related problem for an approximated kernel. The unknown coefficients are recovered by a finite number of measurements of certain time-dependent characteristics of the solution of the direct problem. In [11,12] inverse problems of such a type were studied. These papers deal with the simplified case when the model contains only the relaxation kernel of heat flux. However, a more precise model of a material with thermal memory involves two relaxation kernels contained in basic constitutive relations: kernel of internal energy and heat flux [3,5,8].

In the present paper we study an inverse problem to determine degenerate nonhomogeneous relaxation kernels of internal energy and heat flux by means of a finite number of measurements of temperature in fixed points over the time. As in [11,12] we apply the fixed-point argument in weighted norms adjusted to the problem in the Laplace domain. Due to the structure of the problem, the kernels of internal energy and heat flux are recovered with different level of regularity.

In Section 2 we formulate the direct and inverse problems and in Section 3 apply the Laplace transform to them. In Section 4 we rewrite the transformed problems in the fixed-point form. Sections 5 and 6 contain some auxiliary results for the direct problem. Main existence and uniqueness results for the inverse problem are included in Section 7 of the paper.

2. FORMULATION OF PROBLEM

We consider the heat flow in a rigid nonhomogeneous bar consisting of a material with thermal memory. For a sake of simplicity we assume the rod to be of the unit length. Then, in the linear approximation the heat equation for the bar read as

$$\beta(x)\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial t}\int\limits_0^t n(x,t-\tau)u(x,\tau)\,d\tau=\frac{\partial}{\partial x}\left(\lambda(x)u_x(x,t)\right) \tag{1}$$
$$-\frac{\partial}{\partial x}\int\limits_0^t m(x,t-\tau)u_x(x,\tau)\,d\tau+r(x,t),\quad x\in(0,1),\ t>0.$$

Here u is the temperature of the bar, which is assumed to be zero for $t<0$. Moreover, β is the heat capacity, λ the heat conduction coefficient and r is the heat supply. The model contains two memory kernels n and m, being the relaxation kernels of the integral energy and the heat flux, respectively.

The function $u(x,t)$ is assumed to satisfy the initial conditions

$$u(x,0)=\varphi(x),\quad x\in(0,1) \tag{2}$$

and the Dirichlet boundary conditions

$$u(0,t) = f_1(t),\; u(1,t) = f_2(t), \quad t > 0 \tag{3}$$

with given functions φ on $[0,1]$ and $f_j,\, j = 1,2$ on $[0,\infty)$. Equation (1) with the conditions (2) and (3) form the direct problem for the temperature u.

In an inverse problem we seek for the kernels n and m. We restrict ourselves to the case of the kernels in the following degenerate forms

$$n(x,t) = \sum_{j=1}^{N_1} \nu_j(x) n_j(t), \quad m(x,t) = \sum_{k=1}^{N_2} \mu_k(x) m_k(t)\,, \tag{4}$$

where $\nu_j,\, j = 1,\ldots,N_1,\, \mu_k,\, k = 1,\ldots,N_2$ are given x-dependent functions and $n_j,\, j = 1,\ldots,N_1$, $m_k,\, k = 1,\ldots,N_2$ are unknown time-dependent coefficients. Formulas (4) hold, for instance, when the medium is piecewise continuous, where n_j and m_k are characteristic functions or smooth approximations of characteristic functions of the subdomains of homogeneity. In general case (4) can be interpreted as finite-dimensional approximations of the actual kernels.

We are going to recover the unknowns n_j and m_k by the measurement of the temperature in $N = N_1 + N_2$ different interior points $x_i \in (0,1)$, $i = 1,\ldots,N$, i.e., by the additional conditions

$$u(x_i,t) = h_i(t), \quad i = 1,...,N,\; t > 0\,, \tag{5}$$

where h_i are given functions. Relations (1) - (5) form the *inverse problem* for n and m.

3. APPLICATION OF LAPLACE TRANSFORM

We apply the Laplace transform and designate corresponding images as

$$U(x,p) = \mathcal{L}u = \int_0^\infty e^{-pt} u(x,t)\,dt, \quad \operatorname{Re} p > \sigma, \quad N_j(p) = \mathcal{L}n_j,\; M_k(p) = \mathcal{L}m_k,\; R(x,p) = \mathcal{L}r.$$

Equation (1) takes the form

$$(LU)(x,p) = p\sum_{j=1}^{N_1} N_j(p)\nu_j(x)U(x,p) + \sum_{k=1}^{N_2} M_k(p)\frac{\partial}{\partial x}\left(\mu_k(x)U_x(x,p)\right) - R(x,p) - \beta(x)\varphi(x) \tag{6}$$

where L is a differential operator given by

$$(LU)(x,p) = \frac{\partial}{\partial x}\left(\lambda(x)U_x(x,p)\right) - \beta(x)pU(x,p), \quad x \in (0,1). \tag{7}$$

The boundary conditions (3) are transformed to

$$U(0,p) = F_1(p), \quad U(1,p) = F_2(p), \quad \operatorname{Re} p > \sigma, \quad \sigma \in \mathrm{R} \tag{8}$$

where $F_j(p) = \mathcal{L}f_j$, $j = 1,2$.
Let us denote by $G(x,y,p)$ the Green function of operator L with homogeneous boundary conditions

$$L_y G(x,y,p) = \delta(x,y), \qquad x \in (0,1), y \in (0,1) \tag{9}$$

$$G(x,0,p) = G(x,1,p) = 0\,, \tag{10}$$

where L_y designate the operator L applied with respect to the variable y.
Then the solution of (6) is given by

$$\begin{aligned} U(x,p) &= \sum_{j=1}^{N_1} N_j(p)\int_0^1 G(x,y,p)\nu_j(y)pU(y,p)\,dy \qquad (11)\\ &+ \sum_{k=1}^{N_2} M_k(p)\int_0^1 G(x,y,p)\frac{\partial}{\partial y}\left(\mu_k(y)U_y(y,p)\right)\,dy - F(x,p) \end{aligned}$$

where

$$F(x,p)=\int_0^1 G(x,y,p)\left[\beta(y)\varphi(y)+R(y,p)\right]\,dy+\lambda(0)G_y(x,0,p)F_1(p)-\lambda(1)G_y(x,1,p)F_2(p)\,. \qquad (12)$$

Integrating the second integral of (11) by parts and using (10) we have

$$\begin{aligned}U(x,p)=&\sum_{j=1}^{N_1}pN_j(p)\int_0^1 G(x,y,p)\nu_j(y)U(y,p)\,dy\\&-\sum_{k=1}^{N_2}M_k(p)\int_0^1 G_y(x,y,p)\mu_k(y)U_y(y,p)\,dy-F(x,p).\end{aligned} \qquad (13)$$

Further, differentiating (11) with respect to x and using properties of the Green function it is possible to derive the following equation for $U_x(x,p)$:

$$\begin{aligned}U_x(x,p)\;&=\;\frac{1}{\lambda(x)}\sum_{k=1}^{N_2}M_k(p)\mu_k(x)U_x(x,p)+\sum_{j=1}^{N_1}pN_j(p)\int_0^1 G_x(x,y,p)\nu_j(y)U(y,p)\,dy\\&-\;\sum_{k=1}^{N_2}M_k(p)\int_0^1 G_{xy}(x,y,p)\mu_k(y)U_y(y,p)\,dy-F_x(x,p)\end{aligned} \qquad (14)$$

with F given by (12). Summing up, (13) and (14) form a system of integral equations for functions $U(x,p)$ and $U_x(x,p)$.

4. FIXED-POINT SYSTEMS FOR INVERSE AND DIRECT PROBLEMS

In this section we deduce a fixed-point system for the inverse problem in the Laplace domain and transform further the system for U and U_x.

Applying Laplace transform to additional conditions (5) yields

$$U(x_i,p)=H_i(p),\quad i=1,\ldots,N=N_1+N_2. \qquad (15)$$

First, let us study the behaviour of the equation (13) in the process $\mathrm{Re}p\to\infty$. We suppose a priori that the inverse problem has a solution n_k, m_l satisfying conditions $pN_k(p)\to n_k(0)$ and $\sqrt{p}M_l(p)\to 0$ as $\mathrm{Re}p\to\infty$.

Using these asymptotic relations and the properties of the Green function [12] we obtain from (13) multiplied by p^2 the equalities

$$-\sum_{k=1}^{N_1}n_k(0)\frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)=\lim_{\mathrm{Re}p\to\infty}p^2\left[H_i(p)+F(x_i,p)\right],\quad i=1,\ldots,N=N_1+N_2. \qquad (16)$$

This is a system for initial values $n_k(0)$ of the unknowns n_k. We suppose that the rank of the matrix and the extended matrix of this system equals N_1 implying the existence of unique solution of (16).

Note that system (13), (14) involves the unknowns in the form $pN_k(p)$ and $M_l(p)$. This suggests that the kernels n_k and m_l can be determined simultaneously with higher smoothness in n_k than in m_l. Therefore we define

$$Q_k(p)\;=\;pN_k(p)-n_k(0)\;=\;\mathcal{L}(n_k') \qquad (17)$$

and derive a fixed-point system for Q_k, M_l. We introduce the matrix $\Gamma=(\gamma_{ik})_{i,k=1,\ldots,N}$ related to this principal part, where

$$\gamma_{ik}\;=\;\begin{cases}\frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i), & k=1,\ldots,N_1,\\ \frac{1}{\beta(x_i)}\left(\mu_{k-N_1}(y)\varphi'(y)\right)'\Big|_{y=x_i}, & k=N_1+1,\ldots,N\end{cases} \qquad (18)$$

and assume $\det\Gamma\neq 0$. Further, we introduce the unified notation for the unknowns

$$Z_k=\begin{cases}Q_k, & k=1,\ldots,N_1,\\ M_{k-N_1}, & k=N_1+1,\ldots,N,\end{cases}\qquad Z=(Z_1,\ldots,Z_N) \qquad (19)$$

We define vanishing with Re$p \to \infty$ functions

$$B^0[Z](x,p) \;=\; pU[Z](x,p) - \varphi(x)\,, B^1[Z](x,p) \;=\; pU_x[Z](x,p) - \varphi'(x)\,, \tag{20}$$

where $U[Z](x,p)$ is the Laplace transform of the Z-dependent solution of the direct problem. Now from (13) in view of (20) we deduce the following fixed-point system for Z:

$$Z \;=\; \Gamma^{-1}\mathcal{F}(Z)\,, \tag{21}$$

where $\mathcal{F}(Z) = (\mathcal{F}_1(Z), \ldots, \mathcal{F}_N(Z))$,

$$\begin{aligned}
\mathcal{F}_i[Z](p) \;&=\; \sum_{k=1}^{N_1} Z_k(p)\left[\int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy\right.\\
&+ \left.\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\right]\\
&+ \sum_{k=N_1+1}^{N} Z_k(p)\left[-\int_0^1 pG_y(x_i,y,p)\mu_{k-N_1}(y)B^1[Z](y,p)\,dy\right.\\
&+ \left.\int_0^1 pG(x_i,y,p)\left(\mu_{k-N_1}(y)\varphi'(y)\right)'\,dy \;+\frac{1}{\beta(x_i)}\left(\mu_{k-N_1}(x)\varphi'(x)\right)'\Big|_{x=x_i}\right]\\
&+ \sum_{k=1}^{N_1} n_k(0)\int_0^1 pG(x_i,y,p)\nu_k(y)B^0[Z](y,p)\,dy + \widehat{\Psi}_i(p), \quad i=1,\ldots,N
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\widehat{\Psi}_i(p) \;&=\; \sum_{k=1}^{N_1} n_k(0)\left[\int_0^1 pG(x_i,y,p)\nu_k(y)\varphi(y)\,dy + \frac{1}{\beta(x_i)}\nu_k(x_i)\varphi(x_i)\right]\\
&- \; p^2\left[H_i(p) + F(x_i,p)\right] + \lim_{\mathrm{Re}q\to\infty} q^2\left[H_i(q) + F(x_i,q)\right].
\end{aligned} \tag{23}$$

For future analysis we need a proper fixed-point system for the quantities $B^0[Z]$ and $B^1[Z]$, too. We deduce this system from (13), (14) using (20). For the function $B^0[Z]$, which in contrast to $B^1[Z]$ doesn't contain a space derivative of U, we need a certain higher regularity in the time variable. To this end we will assume that the free term Φ^0 of $B^0[Z]$ can be decomposed as follows

$$\Phi^0(x,p) = \sum_{k=1}^{N_1} n_k(0)\int_0^1 G(x,y,p)\nu_k(y)\varphi(y)\,dy - pF(x,p) - \varphi(x) = B^{0,0}(x,p) + \widetilde{\Phi}(x,p)\,, \tag{24}$$

where $|B^{0,0}(x,p)| \le \frac{\mathrm{Const}}{|p|}$ and $|\widetilde{\Phi}(x,p)| \le \frac{\mathrm{Const}}{|p|^\alpha}$ with some $\alpha > 1$ for Re$p > \sigma_0$, $x \in [0,1]$ and split $B^0[Z]$ into the sum

$$B^0[Z](x,p) = B^{0,0}(x,p) + B^{0,1}[Z](x,p)\,, \tag{25}$$

where for $B^{0,1}$ we will require that $|B^{0,1}[Z](x,p)| \le \frac{\mathrm{Const}}{|p|^\alpha}$ for Re$p > \sigma_0$, $x \in [0,1]$.

From (13) and (14) in view of (24), (25) we deduce the following fixed-point equation for the vector $B[Z] = (B^{0,1}[Z], B^1[Z])$:

$$B[Z] = A[Z]B[Z] + b[Z]\,, \tag{26}$$

where $A[Z]=(A^0[Z],A^1[Z])$ is the Z-dependent linear operator of B with the components

$$
\begin{aligned}
\left(A^0[Z]B\right)(x,p) &= \sum_{k=1}^{N_1}(Z_k(p)+n_k(0))\int_0^1 G(x,y,p)\nu_k(y)B^{0,1}(y,p)\,dy \\
&- \sum_{k=N_1+1}^{N} Z_k(p)\int_0^1 G_y(x,y,p)\mu_{k-N_1}(y)B^1(y,p)\,dy, \qquad (27)\\
\left(A^1[Z]B\right)(x,p) &= \sum_{k=1}^{N_1}(Z_k(p)+n_k(0))\int_0^1 G_x(x,y,p)\nu_k(y)B^{0,1}(y,p)\,dy \\
&+ \sum_{k=N_1+1}^{N} Z_k(p)\left[\frac{\mu_{k-N_1}(x)}{\lambda(x)}B^1(x,p)-\int_0^1 G_{xy}(x,y,p)\mu_{k-N_1}(y)B^1(y,p)\,dy\right] \qquad (28)
\end{aligned}
$$

and $b[Z]=(b^0[Z],b^1[Z])$ is the Z-dependent B-free term with the components

$$
\begin{aligned}
b^0[Z](x,p) &= \sum_{k=1}^{N_1} Z_k(p)\int_0^1 G(x,y,p)\nu_k(y)\left[B^{0,0}(y,p)+\varphi(y)\right]\,dy \\
&- \sum_{k=N_1+1}^{N} Z_k(p)\int_0^1 G_y(x,y,p)\mu_{k-N_1}(y)\varphi'(y)\,dy+\Phi^{0,1}(x,p), \qquad (29)
\end{aligned}
$$

$$
\begin{aligned}
b^1[Z](x,p) &= \sum_{k=1}^{N_1} Z_k(p)\int_0^1 G_x(x,y,p)\nu_k(y)\left[B^{0,0}(y,p)+\varphi(y)\right]\,dy \\
&+ \sum_{k=N_1+1}^{N} Z_k(p)\Big[\frac{\mu_{k-N_1}(x)\varphi'(x)}{\lambda(x)}-\int_0^1 G_{xy}(x,y,p)\mu_{k-N_1}(y)\varphi'(y)\,dy\Big]+\Phi^1(x,p) \qquad (30)
\end{aligned}
$$

and

$$\Phi^{0,1}(x,p)=\sum_{k=1}^{N_1} n_k(0)\int_0^1 G(x,y,p)\nu_k(y)B^{0,0}(y,p)\,dy+\widetilde{\Phi}(x,p), \qquad (31)$$

$$\Phi^1(x,p)=\sum_{k=1}^{N_1} n_k(0)\int_0^1 G_x(x,y,p)\nu_k(y)\left[B^{0,0}(y,p)+\varphi(y)\right]\,dy-pF_x(x,p)-\varphi'(x). \qquad (32)$$

5. FUNCTIONAL SPACES

To analyse the direct and inverse problems we define the spaces

$$\mathcal{A}_{\gamma,\sigma}=\{V: V(p)\quad \text{is holomorphic on Re}p>\sigma,\ \|V\|_{\gamma,\sigma}<\infty\}\,,\quad \gamma,\sigma\in \mathrm{R} \qquad (33)$$

where

$$\|V\|_{\gamma,\sigma}=\sup_{\mathrm{Re}p>\sigma}|p|^{\gamma}|V(p)|$$

and

$$\left(\mathcal{A}_{\gamma,\sigma}\right)^N=\{V=(V_1,\ldots,V_N): V_k(p)\in\mathcal{A}_{\gamma,\sigma}, k=1,\ldots,N\} \qquad (34)$$

with the norm

$$\|V\|_{\gamma,\sigma}=\sum_{k=1}^{N}\|V_k\|_{\gamma,\sigma}\,,\quad V\in\left(\mathcal{A}_{\gamma,\sigma}\right)^N.$$

We note that $\mathcal{A}_{\gamma,\sigma}\subset\mathcal{A}_{\gamma,\sigma'}$, $\left(\mathcal{A}_{\gamma,\sigma}\right)^N\subset\left(\mathcal{A}_{\gamma,\sigma'}\right)^N$ and $\|\cdot\|_{\gamma,\sigma'}\le\|\cdot\|_{\gamma,\sigma}$ if $\sigma'>\sigma$.

Let α be a real number such that

$$1 < \alpha < \frac{3}{2}\,. \tag{35}$$

Moreover, let $c = (c_1, \dots, c_N)$ be a given vector. We will search the solution $Z = (Z_1, \dots, Z_N)$ of (21) from the space

$$\mathcal{M}_{c,\sigma} = \left\{ Z : Z = \frac{c}{p} + V(p),\ V \in (\mathcal{A}_{\alpha,\sigma})^N \right\}.$$

Furthermore, we introduce the spaces of x- and p-dependent functions

$$\mathcal{B}_{\gamma,\sigma} = \{F(x,p)\ :\ F(x,\cdot) \in \mathcal{A}_{\gamma,\sigma} \text{ for } x \in [0,1]\,,\ F(\cdot,p) \in C[0,1] \text{ for } \mathrm{Re}p > \sigma\} \tag{36}$$

with the norms

$$\|F\|_{\gamma,\sigma} = \max_{0\le x\le 1} \sup_{\mathrm{Re}p>\sigma} |p|^\gamma |F(x,p)|.$$

Let α' be a given number such that

$$\alpha < \alpha' < \frac{3}{2}\,. \tag{37}$$

We are going to solve the equation (26) for the pair $B = \left(B^{0,1}, B^1\right)$ in the space $\mathcal{B}_\sigma = \mathcal{B}_{\alpha',\sigma} \times \mathcal{B}_{1,\sigma}$ with the norm

$$\|B\|_\sigma = \|B^{0,1}\|_{\alpha',\sigma} + \|B^1\|_{1,\sigma}\,.$$

6. ANALYSIS OF DIRECT PROBLEM

Let us assume

$$\left.\begin{array}{l}
\lambda, \beta \in C^2[0,1], \quad \lambda, \beta > 0;\\
\Phi^0 \text{ given by (24) admits the decomposition (24)}\\
\text{where } B^{0,0} \in \mathcal{B}_{1,\sigma_0} \text{ and } \widetilde{\Phi} \in \mathcal{B}_{\alpha',\sigma_0} \text{ with some } \sigma_0 \ge 1 \text{ and } \alpha, \alpha' \text{satisfying (35), (37);}\\
\Phi^1 \text{ given by (32) belongs to } \mathcal{B}_{1,\sigma_0};\\
\nu_k \in C[0,1],\ k = 1, \dots, N_1\,,\ \ \mu_l \in C^1[0,1],\ l = 1, \dots, N_2;\ \ \varphi \in C^2[0,1].
\end{array}\right\} \tag{38}$$

Lemma 1. *Let the assumptions* (38) *hold. If* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ *then the vector function* $b[Z] = (b^0[Z], b^1[Z])$, *given by* (29), (30), *belongs to* $\mathcal{B}_{\sigma_0}$ *and satisfies the estimate*

$$\|b[Z]\|_\sigma \quad \le \quad C_1 \left[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha'}} \left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}} \right) \right] \tag{39}$$

with any $\sigma \ge \sigma_0$, *where* C_1 *is a constant and* $|c| = \sum\limits_{k=1}^{N} |c_k|$. *Moreover, for every* $\sigma \ge \sigma_0$ *and* $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$ *the difference* $b\left[Z^1\right] - b\left[Z^2\right]$ *fulfils the estimate*

$$\left\| b[Z^1] - b[Z^2] \right\|_\sigma \le C_2 \frac{1}{\sigma^{\alpha-\alpha'+\frac{1}{2}}} \left\| V^1 - V^2 \right\|_{\alpha,\sigma} \tag{40}$$

with a constant C_2.

P r o o f. Multiplying (29) by $|p|^{\alpha'}$ and estimating absolute value using properties of the Green function [12], we obtain (39) with $\left\|b^0[Z]\right\|_{\alpha',\sigma}$ in the left side. Next we multiply (30) by $|p|$ and perform similar estimation to obtain (39) with $\left\|b^1[Z]\right\|_{1,\sigma}$ in the left side. This means, that the estimate (39) holds for whole vector $b[Z] = (b^0[Z], b^1[Z])$. Then the components $b^0[Z]$ and $b^1[Z]$ of the vector $b[Z] = b[Z^1] - b[Z^2]$ are expressed by the formulas (29) with $\Phi^{0,1} = 0$ and (30) with $\Phi^1 = 0$, respectively. Using the estimates for the components of $b[Z]$ and observing that $Z = \frac{c}{p} + V$ with $c = 0$ and $V = V^1 - V^2$ we deduce (40). The proof is complete.

Lemma 2. *Let the assumptions* (38) *hold.* If $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ *then the linear operator* $A[Z] = (A^0[Z], A^1[Z])$, *defined by* (27), (28), *is bounded in* $\mathcal{B}_\sigma$ *and satisfies the estimate*

$$\|A[Z]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \le C_3 \left[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac{1}{2}}} \right] \tag{41}$$

for any $\sigma \geq \sigma_0$ with a constant C_3. Moreover, taking $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$, the estimate for difference

$$\left\| \left(A[Z^1] - A[Z^2] \right) \right\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \leq C_4 \frac{1}{\sigma^\alpha} \left\| V^1 - V^2 \right\|_{\alpha,\sigma} \tag{42}$$

holds for any $\sigma \geq \sigma_0$ with a constant C_4.

P r o o f. Multiplying (27) by $|p|^{\alpha'}$ and estimating absolute value we deduce

$$\left\| A^0[Z]B \right\|_{\alpha',\sigma} \leq C_8 \left[\frac{|c|}{\sigma^{\frac{5}{2}-\alpha'}} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\frac{3}{2}-\alpha'+\alpha}} + \frac{|n(0)|}{\sigma} \right] \|B\|_\sigma , \quad \sigma \geq \sigma_0 , \tag{43}$$

where C_8 is a constant. similar way, multiplying (28) by $|p|$, we obtain

$$\left\| A^1[Z]B \right\|_{1,\sigma} \leq C_9 \left[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{|n(0)|}{\sigma^{\alpha'-\frac{1}{2}}} \right] \|B\|_\sigma , \quad \sigma \geq \sigma_0 \tag{44}$$

with a constant C_9. These two estimates together prove that $A[Z]$ is bounded in $\mathcal{B}_\sigma$ and satisfies estimate (41). It remains to prove (42). Denoting $Z = Z^1 - Z^2$ the components $A^0[Z]$ and $A^1[Z]$ of the vector $A[Z] = A[Z^1] - A[Z^2]$ are expressed by the formulas (27) and (28), respectively, containing $n_k(0) = 0$. Using the estimates (43), (44) and observing that $Z = \frac{c}{p} + V$ with $c = 0$ and $V = V^1 - V^2$ we deduce (42). Lemma is proved.

Due to Lemmas 1, 2 and the contraction principle equation (26) has a unique solution $B = B[Z] \in \mathcal{B}_\sigma$ provided $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$ and $\sigma \geq \sigma_0$ satisfy the relation

$$\eta(Z,\sigma) := \frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac{1}{2}}} < \frac{1}{C_3} . \tag{45}$$

Furthermore, from (26) we have $\|B[Z]\|_\sigma \leq \left(1 - \|A[Z]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma}\right)^{-1} \|b[Z]\|_\sigma$. This in view of (39), (41) and (45) yields the estimate

$$\|B[Z]\|_\sigma \leq C_1 \left\{ 1 - C_3 \left[\frac{|c|}{\sigma} + \frac{\|V\|_{\alpha,\sigma}}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac{1}{2}}} \right] \right\}^{-1} \left[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha'}} \left(|c| + \frac{\|V\|_{\alpha,\sigma}}{\sigma^{\alpha-1}} \right) \right] \tag{46}$$

for the solution of (26).

Next let us find an estimate for $B[Z^1] - B[Z^2]$. Let $\sigma \geq \sigma_0$ and $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2$ be such that (45) is valid for V replaced by V^1 and V^2 i.e. $\eta(Z^j,\sigma) < \frac{1}{C_3}$, $j = 1,2$. Subtracting equation (26) for $Z = Z^2$ from the corresponding equation for $Z = Z^1$ we have

$$B[Z^1] - B[Z^2] = A[Z^2] \left(B[Z^1] - B[Z^2] \right) + \left(A[Z^1] - A[Z^2] \right) B[Z^1] + b[Z^1] - b[Z^2].$$

This implies

$$\|B[Z^1] - B[Z^2]\|_\sigma \leq \left(1 - \|A[Z^2]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \right)^{-1} \left[\|A[Z^1] - A[Z^2]\|_{\mathcal{B}_\sigma \to \mathcal{B}_\sigma} \|B[Z^1]\|_\sigma + \|b[Z^1] - b[Z^2]\|_\sigma \right] .$$

Using in this relation the estimates (40) - (46) we obtain

$$\left\| B[Z^1] - B[Z^2] \right\|_\sigma \leq C_5 \left\| V^1 - V^2 \right\|_{\alpha,\sigma} \tag{47}$$

with a constant C_5. Summing up, we have proved the following theorem.

Theorem 1. *Let the assumptions* (38) *hold. Then there exists a constant $C_3 > 0$ depending on the data of equation* (26) *such that for any $\sigma \geq \sigma_0$ and $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma}$, satisfying the inequality* (45), *equation* (26) *has a unique solution $B[Z] = \left(B^{0,1}[Z], B^1[Z] \right)$ in $\mathcal{B}_\sigma$. This solution satisfies estimate* (46). *Moreover, for every $\sigma \geq \sigma_0$ and $Z^1 = \frac{c}{p} + V^1$, $Z^2 = \frac{c}{p} + V^2 \in \mathcal{M}_{c,\sigma}$ such that $\eta(Z^j,\sigma) < \frac{1}{C_3}$, $j = 1,2$, the difference $B[Z^1] - B[Z^2]$ fulfils estimate* (47).

7. EXISTENCE AND UNIQUENESS FOR INVERSE PROBLEM

In this section we study the inverse problem in the fixed-point form (21) in the Laplace domain and thereupon infer a result for the inverse problem (1) - (5) in the time domain.

Due to the decomposition (25) the full Z-free term of the operator $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_N)$ given by (22) is $\Psi = (\Psi_1, \ldots, \Psi_N)$, where

$$\Psi_i(p) = \widehat{\Psi}_i(p) + \sum_{k=1}^{N_1} n_k(0) \int_0^1 pG(x_i, y, p)\nu_k(y) B^{0,0}(y,p)\, dy. \tag{48}$$

and $\widehat{\Psi}_i$ is defined in (23)

Theorem 2. *Assume that* (38) *holds and*

$$\nu_k \in C^1[0,1]\,, \quad k = 1, \ldots, N_1\,; \qquad \mu_l \in C^2[0,1]\,, \quad l = 1, \ldots, N_2\,; \qquad \varphi \in C^3[0,1]. \tag{49}$$

Moreover, let $\det \Gamma \neq 0$ *for* Γ *given by* (18) *and*

$$\Psi \;=\; \frac{d}{p} + Y \in \mathcal{M}_{d,\sigma_0} \tag{50}$$

with some $d \in \mathrm{R}^N$*. Then there exists* $\sigma_1 \geq \sigma_0$ *such that equation* (21) *has a unique solution* $Z = \frac{c}{p} + V \in \mathcal{M}_{c,\sigma_1}$*. Here* $c = \Gamma^{-1} d$*.*

P r o o f. Setting $c = \Gamma^{-1} d$ and observing (25), problem (21) in $\mathcal{M}_{c,\sigma}$ is equivalent to the following equation for V in $(\mathcal{A}_{\alpha,\sigma})^N$

$$V = F(V)\,, \tag{51}$$

where $F = \Gamma^{-1}\widetilde{F}$. $\widetilde{F}$ is deduced from (22) taking $Z = \frac{c}{p} + V$ and using (50).

We will prove the assertion of theorem using the fixed-point argument in the following balls:

$$D_{\alpha,\sigma}(\rho) = \left\{ V \in (\mathcal{A}_{\alpha,\sigma})^N : \|V\|_{\alpha,\sigma} \leq \rho \right\}.$$

Multiplying $\widetilde{F}(V)$ by $|p|^\alpha$ and estimating using properties of the Green function [12] we obtain

$$\left\|\widetilde{F}(V)\right\|_{\alpha,\sigma} \leq C_6 \left\{ \left(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{\|V\|_{\alpha,\sigma}}{\sqrt{\sigma}} \right) (\|B\,[Z]\|_\sigma + 1) + \frac{1}{\sigma^{\alpha'-\alpha}} \|B\,[Z]\|_\sigma \right\} + \|Y\|_{\alpha,\sigma_0}\,, \quad \sigma \geq \sigma_0 \tag{52}$$

with a constant C_6 depending on the data of the problem.

Further, let us suppose that $V \in D_{\alpha,\sigma}(\rho)$, where σ and ρ satisfy

$$\eta_0(\rho,\sigma) \;:=\; \frac{|c|}{\sigma} + \frac{\rho}{\sigma^\alpha} + \frac{1}{\sigma^{\alpha'-\frac{1}{2}}} \;<\; \frac{1}{C_3} \tag{53}$$

and $\sigma \geq \sigma_0$. Then (45) holds, hence we can apply estimate (46) of Theorem 1 for $\|B[Z]\|_\sigma$. Plugging (46) into (52) and estimating $\|V\|_{\alpha,\sigma}$ by ρ we have

$$\begin{aligned} \left\|\widetilde{F}(V)\right\|_{\alpha,\sigma} &\leq C_6 \left\{ \left(\frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{1}{\sigma^{\alpha'-\alpha}} + \frac{\rho}{\sqrt{\sigma}} \right) C_1 \left[1 - \eta_0(\rho,\sigma) C_3\right]^{-1} \right. \\ &\times \left. \left[1 + \frac{1}{\sigma^{\frac{3}{2}-\alpha'}} \left(|c| + \frac{\rho}{\sigma^{\alpha-1}} \right) \right] + \frac{|c|}{\sigma^{\frac{3}{2}-\alpha}} + \frac{\rho}{\sqrt{\sigma}} \right\} + \|Y\|_{\alpha,\sigma_0}\,. \end{aligned} \tag{54}$$

From (54) due to the equality $F = \Gamma^{-1}\widetilde{F}$ we see that for every $\rho > \rho_0 := \left|\Gamma^{-1}\right| \|Y\|_{\alpha,\sigma_0}$ there exists $\sigma_2 = \sigma_2(\rho) \geq \sigma_0$ such that the inequalities $\eta_0(\rho,\sigma) < 1/C_3$ and $\|FV\|_{\alpha,\sigma} \leq \rho$ hold for any $\sigma \geq \sigma_2(\rho)$. Consequently,

$$F : D_{\alpha,\sigma}(\rho) \to D_{\alpha,\sigma}(\rho) \quad \text{for} \quad \rho > \rho_0 \quad \text{and} \quad \sigma \geq \sigma_2(\rho). \tag{55}$$

Next, we prove that F is a contraction. To this end from $\widetilde{F}(V)$ with $Z = \frac{c}{p} + V$ and $\widetilde{Z} = \frac{c}{p} + \widetilde{V}$ we derive the expression of $\left(\widetilde{F}(V) - \widetilde{F}(\widetilde{V})\right)_i(p)$. Performing with this expression similar operations as above in deriving (52) we obtain the estimate

$$\|\widetilde{F}(V) - \widetilde{F}(\widetilde{V})\|_{\alpha,\sigma} \leq C_7 \frac{1}{\sqrt{\sigma}} \|V - \widetilde{V}\|_{\alpha,\sigma}\,.$$

The coefficient of $\|V-\widetilde{V}\|_{\alpha,\sigma}$ on the right-hand side of this estimate approaches zero as $\sigma\to\infty$ for a fixed $\rho>0$. Hence, for every $\rho>0$ there exists $\sigma_3=\sigma_3(\rho)\ge\sigma_0$, such that the inequality $\eta_0(\rho,\sigma)<1/C_3$ holds and $F=\Gamma^{-1}\widetilde{F}$ is a contraction in the ball $D_{\alpha,\sigma}(\rho)$ for $\rho>0$ and $\sigma\ge\sigma_3(\rho)$. This together with (55) shows that equation (51) has a unique solution V in every ball $D_{\alpha,\sigma}(\rho)$, where $\rho>\rho_0$ and $\sigma\ge\sigma_4(\rho)=\max(\sigma_2(\rho);\sigma_3(\rho))$. This proves the existence assertion of theorem with $\sigma_1=\sigma_4(2\rho_0)$.

It remains to prove that the solution of (51) is unique in the whole space $(\mathcal{A}_{\alpha,\sigma_1})^N$. Suppose that (51) has two solutions V^1 and V^2 in $(\mathcal{A}_{\alpha,\sigma_1})^N$. Let us define $\bar\rho:=\max(2\rho_0;\left\|V^1\right\|_{\alpha,\sigma_1};\left\|V^2\right\|_{\alpha,\sigma_1})$ and $\bar\sigma:=\max(\sigma_1;\sigma_4(\bar\rho))$. Then we have $\|V^j\|_{\alpha,\sigma_1}\le\bar\rho$, $j=1,2$. Since the norm $\|.\|_{\alpha,\sigma}$ is non-increasing with respect to σ and $\bar\sigma\ge\sigma_1$, from this relation we derive $\|V_j\|_{\alpha,\bar\sigma}\le\bar\rho$ which implies $V^j\in D_{\alpha,\bar\sigma(\bar\rho)}$, $j=1,2$. But due to $\bar\rho>\rho_0$ and $\bar\sigma\ge\sigma_4(\bar\rho)$, the uniqueness in the ball $D_{\alpha,\bar\sigma(\bar\rho)}$ has already been shown. Thus, $V^1=V^2$. Theorem 2 is proved.

Finally, applying the well-known results about the invertibility of the Laplace transform [13] we deduce the following corollary from Theorem 2.

Corollary 1. *Let the rank conditions hold yielding the unique initial values $n_j(0)$ for the unknowns n_j, $k=1,\dots,N_1$ from system* (16). *Moreover, let the assumptions of Theorem* 2 *be satisfied for the functions λ_k, μ_l, φ and the quantities Φ^0, Φ^1, Ψ given by formulas* (24), (32), (48) *via* (24), (12) *in terms of the Laplace transforms R, F_1, F_2, H_i of the data of inverse problem* (1) - (5).

Then inverse problem (1) - (5) *has the unique solution with n_j and m_k of the form*

$$n_j(t) \;=\; n_j(0)+c_jt+\frac{1}{2\pi i}\int_0^t\int_{\xi-i\infty}^{\xi+i\infty}e^{\tau p}Z_j(p)dp\,d\tau\,,\quad k=1,\dots N_1\,,$$

$$m_k(t) \;=\; c_{k-N_1}+\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty}e^{tp}Z_{k-N_1}(p)dp\,d\tau\,,\quad k=1,\dots N_2$$

where $c=(c_1,\dots,c_N)\in\mathrm{R}^N$, $Z=(Z_1,\dots,Z_N)\in(\mathcal{A}_{\alpha,\sigma_1})^N$, $N=N_1+N_2$. The functions n_j are continuously differentiable and m_k are continuous for $t\ge0$. Moreover, $n_j'(0)=c_k$, $j=1,\dots,N_1$ and $m_k(0)=c_{k-N_1}$, $k=1,\dots,N_2$.

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